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An upper bound on Jacobi polynomials

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Abstract

Let $\mathbf{P}_k^{(\alpha,\beta)}(x)$ be an orthonormal Jacobi polynomial of degree k. We will establish the following inequality:

$$\max_{x\in[\delta_{-1},\delta_1]} \sqrt{(x-\delta_{-1})(\delta_1-x)}\,(1-x)^{\alpha}(1+x)^{\beta}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 < \frac{3\sqrt{5}}{5},$$

where $\delta_{-1} < \delta_1$ are appropriate approximations to the extreme zeros of $\mathbf{P}_k^{(\alpha,\beta)}(x)$. As a corollary we confirm, even in a stronger form, T. Erdélyi, A.P. Magnus and P. Nevai conjecture [T. Erdélyi, A.P. Magnus, P. Nevai, Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994) 602–614] by proving that

$$\max_{x\in[-1,1]} (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 < 3\alpha^{1/3}\left(1+\frac{\alpha}{k}\right)^{1/6}$$

in the region $k\geqslant 6$, $\alpha\geqslant\beta\geqslant\frac{1+\sqrt{2}}{4}$.

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1. Introduction

In this paper we will use bold letters for orthonormal polynomials versus regular characters for orthogonal polynomials in the standard normalization [12].

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One of the most surprising and profound features of many families of orthogonal polynomials is their equioscillatory behaviour. This phenomenon has been discovered by G. Szegö who proved that for a vast class of weights the function $\sqrt{1-x^2}\,\mathcal{W}(x)\mathbf{p}_i^2(x)$, asymptotically, for $i \to \infty$, equioscillates between $\pm\frac{2}{\pi}$. Here, $\{\mathbf{p}_i(x)\}$ is a family of orthonormal polynomials of degree i orthogonal with respect to the weight function $\mathcal{W}(x)$ on $[-1, 1]$ [12].

A powerful theory developed for exponential weights $\mathcal{W} = e^{-Q(x)}$ by Levin and Lubinsky [10] shows that under some smoothness assumptions on Q,

$$\max_I \left|\sqrt{|(x-a_{-1})(a_1-x)|}\mathcal{W}(x)\mathbf{p}_i^2(x)\right| < C, \tag{1}$$

where the constant C is independent of i and $a_{\pm 1} = a_{\pm 1}(i)$ are Mhaskar–Rahmanov–Saff numbers for Q. Recently results of this type has been obtained for the Laguerre-type exponential weights $x^{2\rho}e^{-2Q(x)}$ [3,9].

It seems that in many cases (1) is sharp, which means that under an appropriate scaling, the envelope of the function $\sqrt{|(x-\delta_{-1})(\delta_1-x)|}\mathcal{W}(x)\mathbf{p}_i^2(x)$, where $\delta_{\pm 1} = \delta_{\pm 1}(i)$ are certain approximations to the extreme zeros of p_i, is almost independent of i and has a plateau in the oscillatory region with rapidly decaying slopes outside.

Nevertheless, to date we do not possess even a vague picture which could help us to expect such a behaviour. Even the classical orthogonal polynomials are not properly covered by the general theory. For example, the two-sided analogue of (1) with explicit constants and valid independently of the degree and all the parameters involved are known for Hermite [5] and Laguerre [6] polynomials. A similar and, in a sense, best possible upper bound was also given for the Bessel function [8]. However, it is not known whether this is a peculiar property of the hypergeometric function or a manifestation of a more general phenomenon.

Surprisingly enough, the first non-asymptotic inequality of this type was obtained in the seemingly most complicated Jacobi case. Let

$$M_k^{\alpha,\beta}(x) = (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2.$$

Erdélyi et al. [2] proved that for $k \geqslant 0,\ \alpha, \beta \geqslant -\frac{1}{2}$,

$$\mathcal{M}_k^{\alpha,\beta} = \max_{x\in[-1,1]} M_k^{\alpha,\beta}(x) \leqslant \frac{2e\left(2+\sqrt{\alpha^2+\beta^2}\right)}{\pi}. \tag{2}$$

They also observed that $\mathcal{M}_0^{\alpha,\beta} = \frac{\sqrt{\alpha+\beta}}{2\pi}(1+o(1))$ for large α, β and conjectured that the real order of $\mathcal{M}_k^{\alpha,\beta}$ is $O\left(\alpha^{1/2}\right)$.

Inequality (2) was improved to

$$\mathcal{M}_k^{\alpha,\beta} = O\left(\alpha^{2/3}\left(1+\frac{\alpha}{k}\right)^{1/3}\right), \quad \alpha \geqslant \beta,$$

in [4], where a stronger conjecture was suggested,

$$\mathcal{M}_k^{\alpha,\beta} = O\left(\alpha^{1/3}\left(1+\frac{\alpha}{k}\right)^{1/6}\right). \tag{3}$$

Under the classical restrictions $-\frac{1}{2}\leqslant\alpha,\ \beta\leqslant\frac{1}{2}$, much sharper inequalities implying

$$\mathcal{M}_k^{\alpha,\beta}\leqslant\frac{2}{\pi}+O\left(\frac{1}{k}\right),$$

are known [1,11].

Recently, the author proved [7] that in the ultraspherical case for $\alpha\geqslant\frac{1}{2}$,

$$\max_{|x|\leqslant\delta}\sqrt{\delta^2-x^2}\,(1-x)^{\alpha}(1+x)^{\beta}\mathbf{P}_k^{(\alpha,\alpha)}(x)<\frac{2\nu}{\pi}\left(1+\frac{1}{2(k+\alpha)^2}\right),$$

where

$$\delta=\sqrt{1-\frac{4\alpha^2-1}{(2k+2\alpha+1)^2-4}},\qquad \nu=\begin{cases}1, & k\geqslant 4 \text{ even},\\ 228, & k\geqslant 3 \text{ odd}.\end{cases}$$

This implies that for $\alpha\geqslant\frac{1+\sqrt{2}}{4}$, $k\geqslant 6$,

$$\mathcal{M}_k^{\alpha,\alpha}<21\alpha^{1/3}\left(1+\frac{\alpha}{k}\right)^{1/6}.$$

The aim of this paper is to establish similar results in the general case. In particular, we will confirm the above conjectures under some mild restrictions on α, β and k. Namely we prove the following:

Theorem 1. *Let*

$$Z_k^{\alpha,\beta}(x)=\sqrt{\sqrt{(x-\delta_{-1})(\delta_1-x)}(1-x)^{\alpha}(1+x)^{\beta}}\,\mathbf{P}_k^{(\alpha,\beta)}(x), \tag{4}$$

where

$$\delta_{\pm 1}=\frac{\beta^2-\alpha^2\pm\sqrt{(2k+1)(2k+2\alpha+1)(2k+2\beta+1)(2k+2\alpha+2\beta+1)}}{(2k+\alpha+\beta+1)^2}.$$

Then

$$\max_{x\in[\delta_{-1},\delta_1]}|Z_k^{\alpha,\beta}(x)|<5^{-1/4}\sqrt{3} \tag{5}$$

provided $k\geqslant 1$, $\alpha\geqslant\beta\geqslant 0$.

Theorem 2.

$$\mathcal{M}_k^{\alpha,\beta}<3\alpha^{1/3}\left(1+\frac{\alpha}{k}\right)^{1/6} \tag{6}$$

provided $k\geqslant 6$, $\alpha\geqslant\beta\geqslant\frac{1+\sqrt{2}}{4}$.

Notice that the interval $[\delta_{-1},\delta_1]\subset[-1,1]$ defined in Theorem 1 is large enough and contains, for example, all the zeros of $P_k^{(\alpha,\beta)}(x)$. Probably, similar results hold for $\alpha,\ \beta\geqslant-\frac{1}{2}$. Notice also that the assumption $\alpha\geqslant\beta$ does not impose any farther restrictions as $P_k^{(\alpha,\beta)}(x)=(-1)^k P_k^{(\beta,\alpha)}(-x)$.

We believe that the bounds of Theorems 1 and 2 are sharp up to a multiplicative factor. This means that there is a constant c such that, say, under the assumption of Theorem 1, the inequality $|Z_k^{\alpha,\beta}(x)| > c$ holds in a point between any two consecutive zeros of $P_k^{(\alpha,\beta)}(x)$. Furthermore, we suggest the following stronger conjecture:

Conjecture 1.

$$\max_{x\in[\delta_{-1},\delta_1]} \sqrt{|(x-\delta_{-1})(\delta_1-x)|}(1-x)^{\alpha}(1+x)^{\beta}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 = \frac{2}{\pi}(1+o(1))$$

provided $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, *and* $(k+\alpha)\to\infty$.

There is a good reason to believe that $P_k^{(\alpha,\beta)}(x)$ lives on $[\delta_{-1},\delta_1]$, namely:

Conjecture 2.

$$\int_{\delta_{-1}}^{\delta_1}(1-x)^{\alpha}(1+x)^{\beta}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 = 1-o(1)$$

for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, *and* $(k+\alpha)\to\infty$.

Let us outline the proof of Theorems 1 and 2. It is not difficult to find a pointwise upper bound on $P_k^{(\alpha,\beta)}(x)$ in the bulk of the oscillatory region and one such a bound has been already given in [2]. In Section 3 we establish a similar inequality which will be enough for our purposes. Unfortunately, it seems that estimates of this type diverge or become very poor in the transition region around the extreme zeros. The damping factor $\sqrt{(x-\delta_{-1})(\delta_1-x)}$ in (4) is needed to move the global extremum into the oscillatory region. To prove this we use the so-called Sonin's function $S(f(x);x)$ which may be viewed as an envelope of $f(x)$. Then it is a matter of simple algebra to show that for the above choice of $\delta_{\pm 1}$ all local extrema of $\left(Z_k^{\alpha,\beta}(x)\right)^2$ lie on a curve with the unique maximum inside a proper subinterval of $[\delta_{-1},\delta_1]$. This will be accomplished in Section 2. Theorem 1 will be proven in the last section. Passage to Theorem 2 is quite straightforward since all the maxima of $M_k^{\alpha,\beta}(x)$ belong to $[\delta_{-1}+\varepsilon,\delta_1-\varepsilon]$, where $\varepsilon = O\left((k+\alpha)^{-2/3}\right)$. This follows from the results of [4].

2. Unimodality of Sonin's function

Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ are orthogonal polynomials on $[-1,1]$ with the weight function $\mathcal{W}(x)=(1-x)^{\alpha}(1+x)^{\beta}$, $\alpha,\beta>-1$, satisfying the following differential equation

$$(1-x^2)y''+(\beta-\alpha-(\alpha+\beta+2)x)y'+k(k+\alpha+\beta+1)y,\quad y=P_k^{(\alpha,\beta)}(x). \tag{7}$$

In the standard normalization their norm $\mathbf{h}_k$ is given by

$$\mathbf{h}_k^2=\frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}. \tag{8}$$

To avoid unnecessary discussion of some degenerate cases in the sequel we will assume that

$$\alpha > \beta > 0. \tag{9}$$

Theorems 1 and 2 for $\alpha = \beta$, as well as for $\beta = 0$, follow as obvious limiting cases.

To simplify otherwise messy expressions we will use the following notation:

$$\eta = \alpha - \beta, \quad \sigma = \alpha + \beta, \quad r = 2k + \alpha + \beta + 1, \tag{10}$$

$$q = \eta/r, \quad s = \sigma/r;$$

and also their trigonometric counterparts:

$$q = \sin\omega, \quad s = \sin\tau. \tag{11}$$

Thus, (9) yields

$$0 < q < s < 1, \quad 0 < \omega < \tau < \frac{\pi}{2}. \tag{12}$$

Using this notation we can rewrite $\delta_{\pm 1}$ defined in Theorem 1 as follows: $\delta_j = j\cos(\tau + j\omega)$, $j = \pm 1$.

We also introduce the function

$$d(x) = (x - \delta_{-1})(\delta_1 - x) = 1 - q^2 - s^2 - 2qsx - x^2.$$

We will see that in some respects the new variables q, s and r are more natural than α, β and k.

We start with the following simple lemma established in [7]. The proof is straightforward and is given here for self-completeness.

Given a real function $f(x)$, Sonin's function $S = S(f; x)$ is $S = f^2 + \psi(x)f'^2$, where $\psi(x) > 0$ on an interval $\mathcal{I}$ containing all local extrema of f. Thus, they lie on S, and if S is unimodal we can locate the global one.

Lemma 3. *Suppose that a function f satisfies on an interval $\mathcal{I}$ the Laguerre inequality*

$$f'^2 - ff'' > 0, \tag{13}$$

and a differential equation

$$f'' - 2A(x)f' + B(x)f = 0, \tag{14}$$

where $A \in \mathbb{C}$, $B \in \mathbb{C}^1$, and B has at most two zeros on $\mathcal{I}$. Define Sonin's function by

$$S(f; x) = f^2 + \frac{f'^2}{B},$$

then all the local maxima of f in $\mathcal{I}$ are in the interval defined by $B(x) > 0$, and

$$\operatorname{sgn}\left(\frac{d}{dx}S(f; x)\right) = \operatorname{sgn}(4AB - B').$$

Proof. We have $0 < f'^2 - ff'' = f'^2 - 2Aff' + Bf^2$, hence $B(x) > 0$ provided $f' = 0$. Finally,

$$\frac{d}{dx}\left(f^2 + \frac{f'^2}{B}\right) = \frac{4AB - B'}{B^2} f'^2(x), \tag{15}$$

and $B(x) \neq 0$ in an interval containing all the extrema of f on $\mathcal{I}$. $\square$

Let us make a few remarks concerning the Laguerre inequality (13). Usually it is stated for hyperbolic polynomials, that is, real polynomials with only real zeros, and their limiting case, the so-called Polya–Laguerre class of functions. In fact, it holds in a more general situation. Let $L(f) = f'^2 - ff''$. Defining $\mathcal{L} = \{f(x) : L(f) \geqslant 0\}$, we observe that $\mathcal{L}$ is closed under linear transformations $x \to ax + b$. Moreover, since

$$L(fg) = f^2 L(g) + g^2 L(f),$$

$\mathcal{L}$ is closed under multiplication. Therefore, $L(x^\alpha) = \alpha x^{2\alpha-2}$ yields, in particular, the polynomial case. Many more examples may be obtained by $L\left(e^f\right) = -e^{2f} f''$. For our purposes it is important that (13) holds for the function $Z_k^{\alpha,\beta}(x)$ defined by (4), provided $-1 \leqslant \delta_{-1} < x < \delta_1 \leqslant 1$, and $\alpha, \beta \geqslant 0$.

First of all we shall establish the following claim.

Lemma 4. *Let* $Z(x) = Z_k^{\alpha,\beta}(x)$, $k \geqslant 1$, $\alpha > \beta > 0$. *Then the global maximum of the function* $S(Z; x)$ *on the interval* $[\delta_{-1},\ \delta_1]$ *is attained at a point* $x_0 = -qs - \theta\sqrt{(1-q^2)(1-s^2)}$, *where* $0 < \theta < \frac{2}{3}$.

Since the maximum of $S(Z; x)$ is also a local maximum of Z by (15), as an immediate corollary we obtain:

Theorem 5. *The global maximum of the function* $Z_k^{\alpha,\beta}(x)$, $k \geqslant 1$, $\alpha > \beta > 0$, *is attained at a point* x_0 *defined in Lemma* 4.

To prove Lemma 4 we will need the following explicit expressions. It is easy to check that Z satisfies the following ODE:

$$Z'' - 2AZ' + BZ = 0, \tag{16}$$

where

$$A(x) = -\frac{x^3 + 3qsx^2 + (2q^2 + 2s^2 - 1)x + qs}{2(1-x^2)d(x)},$$

$$B(x) = \frac{d(x)r^2}{4(1-x^2)^2} + \frac{E(x)}{4(1-x^2)\,d^2(x)},$$

$$E(x) = 2\,qsx^3 - (1 - 4q^2 - 4s^2 + q^2s^2)x^2 + 6qsx + 1 - q^4 - s^4 + 3q^2s^2.$$

Calculations yield

$$\begin{aligned} D(x) &= 2(1-x^2)^2 d^3(x)(4AB - B') \\ &= qsx^6 + (4q^2 + 4s^2 - 5q^2s^2 - 1)x^5 + qs(12 - q^2 - s^2 + q^2s^2)x^4 \\ &\quad + 2(1 + q^2 + s^2 - 5q^4 - 5s^4 - 5q^2s^2 + q^4s^2 + q^2s^4)x^3 \\ &\quad - qs(7 + 10q^2 + 10s^2 - 4q^2s^2 + q^4 + s^4)x^2 \\ &\quad - (1 + 6q^2 + 6s^2 - 6q^4 - 6s^4 - q^6 - s^6 + 9q^2s^2 + 3q^2s^4 + 3q^4s^2)x \\ &\quad - 3qs(2 - q^2 - s^2 - q^4 - s^4 + 3q^2s^2). \end{aligned}$$

It is quite surprising that this expression does not contain r.

Lemma 4 is an immediate corollary of Lemma 3 and the following claims.

Lemma 6. *$B(x) \neq 0$ for $x \in (\delta_{-1}, \delta_1)$, provided $k \geqslant 0$, $\alpha > \beta > 0$.*

Lemma 7. *$D(x)$ has the only zero x_0 in the interval $[\delta_{-1}, \delta_1]$, provided $k \geqslant 0$, $\alpha > \beta > 0$.*

Lemma 8.

$$x_0 \in \left(-qs - \frac{2}{3}\sqrt{(1-q^2)(1-s^2)}, -qs\right) \subset (\delta_{-1}, \delta_1)$$

provided $k \geqslant 0$, $\alpha > \beta > 0$.

Proof of Lemma 4. By Lemmas 3 and 6,

$$\operatorname{sgn}\left(\frac{d}{dx}S(Z;x)\right) = \operatorname{sgn}(4AB - B') = \operatorname{sgn} D(x).$$

We find

$$D(\delta_j) = -j \cos^3 \tau \cos^3 \omega \sin^4 (\tau + j\omega), \quad j = \pm 1. \tag{17}$$

Hence $D(\delta_{-1}) > 0$, $D(\delta_1) < 0$. Therefore x_0 is the only maximum of $S(Z;x)$ on $[\delta_{-1}, \delta_1]$. □

Let us prove Lemmas 6–8.

Proof of Lemma 6. It is enough to show that $E(x) > 0$ for $\delta_{-1} < x < \delta_1$. Mapping the interval $[\delta_{-1}, \delta_1)$ onto $[0, \infty)$ and simplifying we have

$$\begin{aligned}\frac{2(1+x)^3}{\cos^2\tau \cos^2\omega} E\left(\frac{\delta_{-1}+\delta_1 x}{1+x}\right) &= 10 \sin^2(\tau+\omega)x^3 \\ &\quad + (5 + 2\cos 2\tau + 2\cos 2\omega - \cos 2(\tau+\omega))\,x^2 \\ &\quad + (5 + 2\cos 2\tau + 2\cos 2\omega - \cos 2(\tau-\omega))\,x \\ &\quad + 10 \sin^2(\tau-\omega) > 0.\end{aligned}$$

This completes the proof. □

Let us remind that the discriminant of a polynomial $p = \sum_{i=0}^{n} a_i x^i$, with the zeros $x_1, \ldots, x_n$, is defined by

$$Dis_x\, p = a_n^{2n-2} \prod_{i<j} (x_i - x_j)^2,$$

and can by calculated by the formula

$$Dis_x\, p = (-1)^{n(n-1)/2} a_n^{-1}\, Result_x(p, p'),$$

where $Result_x(p, p')$ states for the resultant of p and p' in x.

It will be a convenient tool to establish positivity of the involved multivariable polynomials. We used Mathematica to find the required resultants.

Proof of Lemma 7. First we map the interval $[\delta_{-1}, \delta_1)$ onto $[0, \infty)$ by considering

$$\frac{(1+x)^6}{(1-q^2)^{3/2}(1-s^2)^{3/2}}\, D\left(\frac{\delta_{-1}+\delta_1 x}{1+x}\right) = \sum_{i=0}^{6}\left(v_i + \sqrt{(1-q^2)(1-s^2}\, u_i)\right) x^i.$$

We will show that the sign pattern of this polynomial is $(+ + +0 - - -)$. Hence by Descartes' rule of signs it has just one positive zero. Since by (17)

$$D(\delta_j) \neq 0, \quad j = \pm 1,$$

this implies the required claim.

We have the following explicit expressions

$$v_0 = 15\left((s^2 - q^2)^2 + 8q^2 s^2 (1 - q^2)(1 - s^2)\right) > 0,$$

$$u_0 = -60qs(q^2 + s^2 - 2q^2 s^2) < 0;$$

$$v_1 = 12\left(4(1 - q^2)(1 - s^2)(q^2 + s^2 + 2q^2 s^2) + 3(s^2 - q^2)^2\right) > 0,$$

$$u_1 = -96qs(1 - q^2 s^2) < 0;$$

$$v_2 = 27(s^2 - q^2)^2 + 40(1 - q^2)(1 - s^2)(q^2 + s^2) + 8(2 - q^2 - s^2 + q^2 s^2) > 0,$$

$$u_2 = -4qs(16 - 7q^2 - 7s^2 - 2q^2 s^2) < 0;$$

$$v_3 = u_3 = 0;$$

$$v_4 = -v_2, \quad u_4 = u_2;$$

$$v_5 = -v_1, \quad u_5 = u_1;$$

$$v_6 = -v_0, \quad u_6 = u_0.$$

Thus, to prove the claim it is left to show that

$$w_i = v_i^2 - (1 - q^2)(1 - s^2)u_i^2 > 0, \quad i = 0, 1, 2.$$

We find

$$w_0 = 225(s^2 - q^2)^4 > 0,$$

$$w_1 = 144(s^2 - q^2)^2\left(8(1 - q^2 s^2)(2 - q^2 - s^2) + (s^2 - q^2)^2\right) > 0,$$

$$w_2 = \left(729 - 864\overline{s} + 160\overline{s}^2\right)\overline{q}^4 + 4\left(135 - 104\overline{s} + 16\overline{s}^2\right)\overline{s}\,\overline{q}^3$$
$$+2\left(11 - 208\overline{s} + 80\overline{s}^2\right)\overline{s}^2\,\overline{q}^2 + 108\,(5 - 8\overline{s})\,\overline{s}^3\,\overline{q} + 729\overline{s}^4,$$

where

$$\overline{q} = \sqrt{1 - q}, \quad \overline{s} = \sqrt{1 - s}, \quad 0 < \overline{s} < \overline{q} \leqslant 1. \tag{18}$$

To demonstrate that $w_2 > 0$ we calculate

$$2^{-32} \cdot 3^{-8}\, Dis_{\overline{q}}\, w_2$$
$$= \overline{s}^{12}(1 - \overline{s})^2(45 - 10\overline{s} + \overline{s}^2)^2(360 - 200\overline{s} - 84\overline{s}^2 - 24\overline{s}^3 - 25\overline{s}^4) \neq 0$$

for $\overline{s} \in (0, 1)$. Therefore w_2 has the same number of real zeros for any $\overline{s} \in (0, 1)$. Since $\overline{s} < \overline{q}$, and for sufficiently small $\overline{s} > 0$, we have $w_2 = 729\overline{q}^4 + O(\overline{s}) > 0$, then w_2 has no real zeros in the region $0 < \overline{s} < \overline{q} < 1$. This completes the proof. $\square$

Proof of Lemma 8. By Lemma 7 it is enough to show that

$$D(-qs) < 0, \tag{19}$$

$$D\left(-qs - \tfrac{2}{3}\sqrt{(1-q^2)(1-s^2)}\right) > 0. \tag{20}$$

We have

$$D(-qs) = -qs(1-q^2)^2(1-s^2)^2\left(5+q^2+s^2-7q^2s^2\right) < 0$$

proving (19).

To prove (20) we use again the change of variables (18) obtaining

$$\frac{729}{5\overline{q}^{3/2}\overline{s}^{3/2}} D\left(-qs - \frac{2}{3}\sqrt{(1-q^2)(1-s^2)}\right) = 6p_1 - \sqrt{\overline{qs}(1-\overline{q})(1-\overline{s})}\,p_2, \tag{21}$$

where

$$p_1 = 1223\overline{q}\,\overline{s}(1-\overline{q})(1-\overline{s}) + 189(\overline{q}-\overline{s})^2 + \overline{q}\,\overline{s}(93-88\overline{qs}) > 0,$$

$$p_2 = 3942\overline{q} + 3942\overline{s} - 6815\overline{qs} > 0.$$

Multiplying (21) by the conjugate yields

$$\begin{aligned} h = 36p_1^2 - \overline{qs}(1-\overline{q})(1-\overline{s})p_2^2 = {} & (9-5\overline{s})(142\,884 - 43\,200\overline{s} - 21\,500\overline{s}^2 + 13\,625\overline{s}^3)\overline{q}^4 \\ & -5(555\,012 - 221\,688\overline{s} + 127\,205\overline{s}^2 - 46\,025\overline{s}^3)\overline{s}\,\overline{q}^3 \\ & +36(87\,988 + 30\,790\overline{s} + 625\overline{s}^2)\overline{s}^2\overline{q}^2 \\ & -4860(571 + 227\overline{s})\overline{s}^3\overline{q} + 128\,5956\overline{s}^4 \end{aligned}$$

and

$$\begin{aligned} & 2^{-4}\cdot 3^{-15}\cdot 5^{-3}\, Dis_{\overline{s}}\, h \\ & = \overline{s}^{12}(1-\overline{s})^2(64\,459\,584 - 111\,438\,880\overline{s} + 47\,706\,875\overline{s}^2)^2(385\,494\,997\,824 \\ & +449\,720\,822\,304\overline{s} - 674\,713\,759\,120\overline{s}^2 + 240\,459\,844\,600\overline{s}^3 - 18\,815\,866\,125\overline{s}^4 \\ & +282\,081\,250\overline{s}^5 - 79\,028\,125\overline{s}^6) \neq 0 \end{aligned}$$

for $\overline{s} \in (0, 1)$.

Since $h > 0$ for $s \to 0^{(+)}$ we conclude that $h > 0$, thus proving (20). □

3. Bounds in the oscillatory region

It will be convenient to introduce the parameter $\rho = 2k + \alpha + \beta = r - 1$ and two functions

$$\mu_j(x) = \frac{\sqrt{(\rho^2-\eta^2)(\rho^2-\sigma^2)} + j(x\rho^2 + \eta\sigma)}{\rho}, \quad j = \pm 1.$$

The Christoffel function in the standard normalization maybe written as

$$\begin{aligned} & \frac{2k!\,\Gamma(k+\alpha+\beta+1)}{(2k+\alpha+\beta)\Gamma(k+\alpha)\Gamma(k+\beta)}\left(P_{k-1}^{(\alpha,\beta)}(x)\frac{d}{dx}P_k^{(\alpha,\beta)}(x) - P_k^{(\alpha,\beta)}(x)\frac{d}{dx}P_{k-1}^{(\alpha,\beta)}(x)\right) \\ & = \sum_{i=0}^{k-1}\frac{(2i+\alpha+\beta+1)i!\,\Gamma(i+\alpha+\beta+1)}{\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}\left(P_i^{(\alpha,\beta)}(x)\right)^2 > 0. \end{aligned}$$

Applying the identity

$$
\begin{aligned}
&2(k+\alpha)(k+\beta)P_{k-1}^{(\alpha,\beta)}(x)\\
&\quad = k\,(\beta-\alpha+(2k+\alpha+\beta)x)\;P_k^{(\alpha,\beta)}(x) + (1-x^2)(2k+\alpha+\beta)\,\frac{d}{dx}P_k^{(\alpha,\beta)}(x)
\end{aligned}
$$

and (7) one finds that

$$
\begin{aligned}
&P_{k-1}^{(\alpha,\beta)}(x)\frac{d}{dx}P_k^{(\alpha,\beta)}(x) - P_k^{(\alpha,\beta)}(x)\frac{d}{dx}P_{k-1}^{(\alpha,\beta)}(x)\\
&\quad = \frac{\rho}{2(\rho^2-\eta^2)}W(x),
\end{aligned}
$$

where

$$
\begin{aligned}
W(x) &= (\rho^2-\sigma^2)y^2 - 4(\eta+\sigma x)yy' + 4(1-x^2)y'^2\\
&= \frac{\mu_{-1}(x)\mu_1(x)}{1-x^2}y^2 + \frac{\left((\eta+\sigma x)y - 2(1-x^2)y'\right)^2}{1-x^2} > 0.
\end{aligned}
$$

Thus, we obtain

$$
y^2 < \frac{1-x^2}{\mu_{-1}(x)\mu_1(x)}W(x) \tag{22}
$$

provided $\mu_{-1}(x)\mu_1(x) > 0$.

To estimate $W(x)$ we will consider the expression

$$
W' - z(x)W
$$

which with help of (7) can be written as a quadratic $U = Ay^2 + Byy' + Cy'^2$ in y and y'. We choose $z(x)$ in such a way that the discriminant of U vanishes, namely

$$
z_j(x) = \frac{d}{dx}\left(\ln\frac{\mu_j(x)}{(1-x)^{\alpha+1}(1+x)^{\beta+1}}\right), \quad j = \pm 1.
$$

For such a choice of z the sign of $U(x)$ coincides with the sign of

$$
C = -j\,\frac{4(1-x^2)\rho}{\mu_j(x)}, \quad j = \pm 1.
$$

Thus, by $W > 0$, in the region

$$
\mathcal{J} = \left(\gamma_{-1}, \gamma_1\right), \quad \gamma_j = \frac{j\sqrt{(\rho^2-\eta^2)(\rho^2-\sigma^2)} - \eta\sigma}{\rho^2}, \tag{23}
$$

defined by

$$
\mu_j(x) > 0, \quad j = \pm 1,
$$

we have

$$
z_{-1}(x) < \frac{W'}{W} < z_1(x).
$$

Solving those inequalities for $x \in \mathcal{J}$, with initial conditions given in a point $\xi \in \mathcal{J}$, one obtains

$$\frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{(1-\xi)^{\alpha+1}(1+\xi)^{\beta+1}}\, W(x) \geqslant \begin{cases} \dfrac{\mu_1(x)}{\mu_1(\xi)} W(\xi), & x \leqslant \xi, \\ \dfrac{\mu_{-1}(x)}{\mu_{-1}(\xi)} W(\xi), & x \geqslant \xi. \end{cases} \tag{24}$$

We will need the value of the following integral.

Lemma 9.

$$\int_{-1}^{1} (1-x)^{\alpha+1}(1+x)^{\beta+1} W(x)\, dx = \frac{(\rho^2-\eta^2)(\rho^2-\sigma^2)}{\rho(\rho-1)}\, \mathbf{h}_k^2. \tag{25}$$

Proof. By

$$\frac{d}{dx} P_k^{(\alpha,\beta)}(x) = \frac{k+\alpha+\beta+1}{2} P_{k-1}^{(\alpha+1,\beta+1)}(x),$$

we have

$$\begin{aligned} &\int_{-1}^{1} (1-x)^{\alpha+1}(1+x)^{\beta+1} \left(P_{k-1}^{(\alpha,\beta)}(x) \frac{d}{dx} P_k^{(\alpha,\beta)}(x) - P_k^{(\alpha,\beta)}(x) \frac{d}{dx} P_{k-1}^{(\alpha,\beta)}(x) \right) dx \\ &\quad = \frac{k+\alpha+\beta+1}{2} \int_{-1}^{1} (1-x)^{\alpha+1}(1+x)^{\beta+1} P_{k-1}^{(\alpha,\beta)}(x) P_{k-1}^{(\alpha+1,\beta+1)}(x)\, dx \\ &\qquad - \frac{k+\alpha+\beta}{2} \int_{-1}^{1} (1-x)^{\alpha+1}(1+x)^{\beta+1} P_k^{(\alpha,\beta)}(x) P_{k-2}^{(\alpha+1,\beta+1)}(x)\, dx. \end{aligned}$$

Now the result follows from the orthogonality relation by the repeated application of the identities

$$(2k+\alpha+\beta) P_k^{(\alpha-1,\beta)}(x) = (k+\alpha+\beta) P_k^{(\alpha,\beta)}(x) - (k+\beta) P_{k-1}^{(\alpha,\beta)}(x),$$

$$(2k+\alpha+\beta) P_k^{(\alpha,\beta-1)}(x) = (k+\alpha+\beta) P_k^{(\alpha,\beta)}(x) + (k+\alpha) P_{k-1}^{(\alpha,\beta)}(x),$$

in order to express $P_i^{(\alpha,\beta)}(x)$ as a sum of Jacobi polynomials with the parameters $\alpha+1, \beta+1$. □

Lemma 10. *For $x \in \mathcal{J}$,*

$$(1-x)^{\alpha+1}(1+x)^{\beta+1} W(x) \leqslant \frac{\rho\sqrt{(\rho^2-\eta^2)(\rho^2-\sigma^2)}}{\rho-1} \mathbf{h}_k^2. \tag{26}$$

Proof. By (24) and (25) we have

$$\begin{aligned} \frac{(\rho^2-\eta^2)(\rho^2-\sigma^2)}{\rho(\rho-1)}\, \mathbf{h}_k^2 &= \int_{-1}^{1} (1-x)^{\alpha+1}(1+x)^{\beta+1} W(x) dx \\ &\geqslant (1-\xi)^{\alpha+1}(1+\xi)^{\beta+1} W(\xi) \left(\int_{\gamma_{-1}}^{\xi} \frac{\mu_1(x)}{\mu_1(\xi)}\, dx + \int_{\xi}^{\gamma_1} \frac{\mu_{-1}(x)}{\mu_{-1}(\xi)}\, dx \right) \\ &= (1-\xi)^{\alpha+1}(1+\xi)^{\beta+1} W(\xi) \frac{\sqrt{(\rho^2-\eta^2)(\rho^2-\sigma^2)}}{\rho^2} \end{aligned}$$

and the result follows. □

Lemma 11. *For $x \in \mathcal{J}$, $k \geqslant 1$,*

$$(1-x)^{\alpha}(1+x)^{\beta}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 < \frac{\sqrt{(1-q^2)(1-s^2)}}{1-q^2-s^2-2qsx-x^2}. \tag{27}$$

Proof. Let us remind that $\rho = r-1$, $\eta = qr$, $\sigma = sr$. Combining (26) with (22) and using

$$\mu_{-1}(x)\mu_1(x) = (1-x^2)\rho^2 - 2\eta\sigma x - \eta^2 - \sigma^2,$$

we obtain the following pointwise bound:

$$(1-x)^{\alpha}(1+x)^{\beta}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 \leqslant \frac{\rho}{\rho-1}\frac{\sqrt{(\rho^2-\eta^2)(\rho^2-\sigma^2)}}{(1-x^2)\rho^2-2\eta\sigma x-\eta^2-\sigma^2} < \frac{\rho}{\rho-1}\frac{\sqrt{(\rho^2-\eta^2)(\rho^2-\sigma^2)}}{(1-q^2-s^2-2qsx-x^2)r^2}.$$

It is left to check that

$$\sqrt{(1-q^2)(1-s^2)} - \frac{\rho}{\rho-1}\frac{\sqrt{(\rho^2-\eta^2)(\rho^2-\sigma^2)}}{r^2} > 0.$$

Multiplying this by the conjugate and writing it down in the variables α, β, $k' = k+1$, one obtains an expression with nonnegative terms only. $\square$

Remark 1. In [2] another pointwise estimate of the order $O\left(\frac{\sqrt{1-x^2}}{1-q^2-s^2-2qsx-x^2}\right)$ was given. The advantage of (27) is that it is stronger for $s = 1-o(1)$, i.e. when $\alpha \gg k$.

Remark 2. Conjecture 2 would imply that (26) and, consequently, (27) are sharp up to a multiplicative constant factor. In turn, this would imply that the bounds of Theorems 1 and 2 are sharp. It is also known that similar results hold for Laguerre polynomials [6].

4. Proof of Theorems 1 and 2

Theorem 1 is an easy corollary of Theorem 5 and Lemma 11. First, we need the following:

Lemma 12. *Suppose that $k \geqslant 1$, $\alpha > \beta > 0$. Then*

$$\left[-qs - \tfrac{2}{3}\sqrt{(1-q^2)(1-s^2)},\, -qs\right] \subset \mathcal{J}.$$

Proof. To prove the claim it is enough to check that

$$p(x) = (x-\gamma_{-1})(\gamma_1 - x) > 0$$

for $x = -qs$ and $x = -qs - \frac{2}{3}\sqrt{(1-q^2)(1-s^2)}$.

Straightforward calculations yield that $\rho^2 r^4 p(-qs)$ written in variables $k' = k+1$, α, β is a polynomial without negative terms.

Similarly,

$$\rho^2 p\left(-qs - \tfrac{2}{3}\sqrt{(1-q^2)(1-s^2)}\right) = r^{-4}p_1(k',\alpha,\beta) + \tfrac{4}{3}qs(2r-1)\sqrt{(1-q^2)(1-s^2)},$$

where $p_1(k',\alpha,\beta)$ is a polynomial in $k' = k+1$, α, β without negative terms. $\square$

Proof of Theorem 1. Let $\mathcal{I} = [-qs - \frac{2}{3}\sqrt{(1-q^2)(1-s^2)}, -qs]$. Since

$$(x-\delta_{-1})(\delta_1 - x) = 1 - q^2 - s^2 - 2qsx - x^2,$$

by Lemma 12, Theorem 5 and (27), we get

$$\max_{x\in[\delta_{-1},\delta_1]} Z^2(x) = \max_{x\in\mathcal{I}} Z^2(x) < \max_{x\in\mathcal{I}} \sqrt{\frac{(1-q^2)(1-s^2)}{1-q^2-s^2-2qsx-x^2}} = \frac{3\sqrt{5}}{5}.$$

This completes the proof of Theorem 1. $\square$

To deduce Theorem 2 from Theorem 1 we will need the following result which has been established in [4].

Theorem 13. *Suppose that $k \geqslant 6$, $\alpha \geqslant \beta \geqslant \frac{1+\sqrt{2}}{4}$. Let x be a point of a local extremum of*

$$(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}\left(P_k^{(\alpha,\beta)}(x)\right)^2.$$

Then $x \in \left(N'_{-1}, N'_1\right)$, where

$$N'_j = j\left(\cos(\tau' + j\omega) - \frac{3}{10}\left(\frac{\sin^4(\tau'+j\omega)}{2\cos\tau'\cos\omega}\right)^{1/3} r^{-2/3}\right), \tag{28}$$

$$\sin\tau' = \frac{\alpha+\beta+1}{2k+\alpha+\beta+1}, \quad 0 < \tau' < \frac{\pi}{2}.$$

We have to restate (28) in terms of τ.

Lemma 14.

$$(N'_{-1}, N'_1) \subset (N_{-1}, N_1) \subset (\delta_{-1}, \delta_1), \tag{29}$$

where

$$N_j = j\left(\cos(\tau + j\omega) - \frac{5}{17}\left(\frac{\sin^4(\tau+j\omega)}{2\cos\tau\cos\omega}\right)^{1/3} r^{-2/3}\right) \tag{30}$$

provided $k \geqslant 6$.

Proof. Since $\sin\tau' = \sin\tau + \frac{1}{r}$, $0 < \tau, \tau' < \frac{\pi}{2}$, then $\tau' > \tau$ and $0 < \tau' \pm \omega < \pi$. Hence,

$$\cos(\tau' + j\omega) < \cos(\tau + j\omega), \quad j = \pm 1,$$

and

$$[N_{-1}, N_1] \subset [-\cos(\tau' - \omega), \cos(\tau' + \omega)] \subset [-\cos(\tau - \omega), \cos(\tau + \omega)] = [\delta_{-1}, \delta_1].$$

We also have for $k \geqslant 6$,

$$\frac{\cos^2\tau'}{\cos^2\tau} = 1 - \frac{2\alpha+2\beta+1}{(2k+1)(2k+2\alpha+2\beta+1)} > 1 - \frac{1}{2k+1} \geqslant \frac{12}{13}.$$

Thus,

$$\frac{\sin^4(\tau' + j\omega)}{\cos\tau'} > \sqrt{\frac{12}{13}}\,\frac{\sin^4(\tau + j\omega)}{\cos\tau},$$

and as $\frac{3}{10} \cdot \left(\frac{12}{13}\right)^{1/6} > \frac{5}{17}$ this implies that $(N'_{-1}, N'_1) \subset (N_{-1}, N_1)$. $\square$

Proof of Theorem 2. To prove Theorem 2 we will bound $M(x) = M_k^{\alpha,\beta}(x)$ by $Z(x)$.

Set $\varepsilon_j = \left(\frac{\sin^4(\tau+j\omega)}{2\cos\tau\cos\omega}\right)^{1/3} r^{-2/3}$. First, we notice that

$$\frac{\varepsilon_j^3}{\cos^3\tau\cos^3\omega} = \frac{(\tan\tau + \tan\omega)^4}{2r^2} < \frac{8\tan^4\tau}{r^2} = \frac{8(\alpha+\beta)^2}{(2k+1)^2(2k+2\alpha+2\beta+1)^2} < \frac{2}{(2k+1)^2} \leqslant \frac{2}{169}.$$

Hence,

$$2\cos\tau\cos\omega - \frac{5\varepsilon_j}{17} > \left(2 - \frac{5}{17}\left(\frac{2}{169}\right)^{1/3}\right)\cos\tau\cos\omega > \frac{27}{14}\cos\tau\cos\omega.$$

By Theorem 13 and Lemma 14 we have

$$\begin{aligned}
\max_{x\in[-1,1]} M(x) &= \max_{x\in[N_{-1},N_1]} M(x) = \max_{x\in[N_{-1},N_1]} \sqrt{\frac{1-x^2}{1-q^2-s^2-2qsx-x^2}}\,Z^2(x)\\
&< \max_{x\in[N_{-1},N_1]} \frac{3}{5}\sqrt{\frac{5(1-x^2)}{1-q^2-s^2-2qsx-x^2}}\\
&< \max_{j=\pm1} \frac{3}{5}\sqrt{\frac{5\sin^2(\tau+j\omega)}{1-q^2-s^2-2qsN_j-N_j^2}}\\
&= \frac{3\sqrt{17}\sin(\tau+j\omega)}{5\sqrt{\varepsilon_j(2\cos\tau\cos\omega - \frac{5}{17}\varepsilon_j)}} < \sqrt{\frac{238}{75}}\,\frac{\sin(\tau+j\omega)}{\sqrt{\varepsilon_j}\cos\tau\cos\omega}\\
&= \sqrt{\frac{238}{75}}\left(\frac{r\sin(\tau+j\omega)}{\cos\tau\cos\omega}\right)^{1/3} = \sqrt{\frac{238}{75}}\,r^{1/3}(\tan\tau+\tan\omega)^{1/3} < \frac{9}{4}(r\tan\tau)^{1/3}\\
&= \frac{9}{4}\left(\frac{(\alpha+\beta)^2(2k+\alpha+\beta+1)^2}{(2k+1)(2k+2\alpha+2\beta+1)}\right)^{1/6} \leqslant \frac{9}{4}\left(\frac{4\alpha^2\,(2k+2\alpha+1)^2}{(2k+1)(2k+4\alpha+1)}\right)^{1/6}\\
&< 3\alpha^{1/3}\left(1+\frac{\alpha}{k}\right)^{1/6}.
\end{aligned}$$

This completes the proof. $\square$

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